INVARIANT SOLUTIONS OF THE EQUATIONS OF A MULTICOMPONENT CHARGED-PARTICLE BEAM

V. A. Syrovoi

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The concept of the invariant-group solution (H-solution) was introduced and a general method for obtaining it was developed in [1–3]. The group properties of the equations of a monoenergetic charged-particle beam with the same value and sign of the specific charge η, assuming univalency of the velocity vector **V**, were studied in [4–6], where all essentially different H-solutions were also constructed. Below, the results of [4–6] are extended to the case of a beam in the presence of a fixed background of density ρ_0 (§1), and also to the case of multivelocity (**V** is an s-valued function) and multicomponent beams (i.e., beams formed by particles of several kinds) (§2). A number of analytic solutions that describe some nonstationary processes in devices with plane, cylindrical, and spherical geometry—among them a continuous periodic solution for a plane diode with a period determined by the background density—are obtained in §1. A transformation that contains arbitrary functions of time and preserves Vlasov's equations is given (§2). The equations studied can be treated as the equations of a rarefied plasma in the magnetohydrodynamic approximation, when the pressure gradients are negligible as compared with forces of electromagnetic origin.

§1. **Invariant solutions of the equations of a beam in the presence of a background.** The equations of a beam in the presence of a background differ from the equations derived in [6] only by additional terms in the Poisson equation

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ik}\frac{\partial\varphi}{\partial x^k}\right)=\rho+\rho_0 \qquad (1.1)$$

Here ρ_0 is the density of the particle background, where $\rho_0 > 0$, if the background is formed by charges of the same sign as the charge of the beam particles, and $\rho_0 < 0$ in the opposite case. The principal group of equations of the beam in the presence of a background will be less general than when $\rho_0 = 0$ [6]; now $X_1 + X_2$ makes sense, but not each of these operators individually

$$X_1+X_2=\mathbf{r}\nabla+\mathbf{V}\nabla_v+2\varphi\frac{\partial}{\partial\varphi} \qquad (1.2)$$

We shall retain the same notation as in [6] for the remaining operators. As usual, in the electrostatic case we have transformations with arbitrary functions of time f, g, h.

We will consider more particularly plane flows.

OPTIMAL SYSTEM OF SINGLE-PARAMETER SUBGROUPS

	1st class	2nd class
1°. X_6	4°. X_3+aX_6	7°. X_5+aX_6
2°. X_4	5°. $(X_1+X_2)+aX_6$	3rd class
3°. X_1+X_2	6°. $X_3+a(X_1+X_2)$	8°. X_7+Y_9

(1.3)

OPTIMAL SYSTEM OF TWO-PARAMETER SUBGROUPS

1st class

1°. $X_5,\ X_6$	4°. $X_1+X_2,\ X_4$
2°. $X_1+X_2,\ X_6$	5°. $X_3+aX_6,\ (X_1+X_2)+bX_6$
3°. $X_3+a(X_1+X_2),\ X_6$	6°. $(X_1+X_2)+aX_6,\ X_5$

(1.4)

2nd class

7°. $X_4+aX_6,\ X_5$	8°. $X_4+Y_7,\ X_5+aX_6$
9°. $X_4+Z_7,\ (X_1+X_2)+aX_6$	

3rd class

10°. $X_8,\ X_9$	12°. $(X_1+X_2)+X_6,\ Z_8$
11°. $X_1+X_2,\ Y_8$	13°. $X_1+X_2,\ X_8+Y_9$

The H-solutions of rank 1 in the table correspond to the enumerated subgroups of the optimal system (1.4).

The H-solutions constructed in subgroups 1°–3° describe stationary flows. For all of the solutions we have

$$\rho=J_5(\xi),\qquad \mathbf{H}=\mathbf{J}_{\varkappa}(\xi)$$

Let us examine some H-solutions of rank 1.

Solution 4° describes the flow between parallel planes y = const.

Table of H-solutions of Rank 1

H-solutions of 1st class

№	conditions on a, b	ξ	**V**	φ
1°		x	$\mathbf{J}_v$	J_4
2°		ψ	$R\mathbf{J}_v$	R^2J_4
3°	$a=-b_2/b_1$	q_1	$e^{b_2q_1}\mathbf{J}_v$	$e^{2b_2q_2}J_4$
4°		t	$y\mathbf{J}_v$	y^2J_4
5°.1	$a=a',\ b=b'$	q_1-t	$R\mathbf{J}_v$	R^2J_4
5°.2	$a=0,\ b\neq 0$	$b\ln R-t$	$R\mathbf{J}_v$	R^2J_4
5°.3	$a\neq 0,\ b=0$	$t-a\psi$	$R\mathbf{J}_v$	R^2J_4
5°.4	$a=0,\ b=0$	t	$R\mathbf{J}_v$	R^2J_4
6°	$a\neq 0$	x^ae^{-t}	$x\mathbf{J}_v$	x^2J_4

$a'=-b_2/(b_1^2+b_2^2),\ b'=b_1/(b_1^2+b_2^2)$

H-solutions of 2nd class ($\mathbf{H}\neq 0,\ a\neq 0$)

№	ξ	**V**	φ
7°	$ax-t$	$\mathbf{J}_v$	J_4
8°	$ay-t$	$\mathbf{J}_v$	$\alpha x+J_4$
9°	y^ae^{-t}	$y\mathbf{J}_v$	$\alpha e^{t/a}x+y^2J_4$

H-solutions of 3rd class ($\mathbf{H}=0$)

№	$\alpha,\ \beta$	ξ	u	v	φ
10°		t	$\frac{f'x}{f}+J_1$	$\frac{g'y}{g}+J_2$	$\frac{f''x^2}{2f}+\frac{g''y^2}{2g}$
11°	$\alpha\neq 0$	t	$\frac{\alpha x}{t}+yJ_1$	yJ_2	$\frac{\gamma x^2}{t^2}+y^2J_4$
12°	$\alpha\neq 0$	ye^{-t}	$\alpha x+yJ_1$	yJ_2	$\frac{\alpha^2x^2}{2}+y^2J_4$
13°.1	$\alpha=1$, $\beta=0$	t	$f'f^{-1}x+$ $+(tx-fy)J_1$	$t^{-1}y+$ $+(tx-fy)J_2$	${}^1/_2f''f^{-1}x^2+$ $+(tx-fy)^2J_4$
13°.2	$\alpha=0$, $\beta=1$	t	$f'f^{-1}x+$ $+(x-fy)J_1$	$(x-fy)J_2$	${}^1/_2f''f^{-1}x^2+$ $+(x-fy)^2J_4$

$\gamma={}^1/_2\alpha(\alpha-1)$

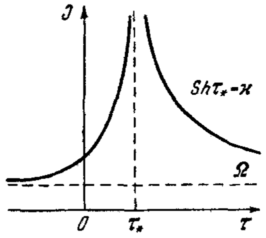

Fig. 1

System (S/H) in the case of a uniform magnetic field H_0 has the following form:

$$J_1' + J_1J_2 = H_0J_2,\ J_2' + J_2^2 = \\ = 2J_4 - H_0J_1,\ J_5' + J_2J_5 = 0, \\ 2J_4 = J_5 + \rho_0 \qquad (1.5)$$

It can be seen that the first of these equations is satisfied when $J_1 = H_0$. Eliminating J_4 and J_5, we obtain

$$J'' + 3JJ' + J^3 + \omega^2 J = 0 \qquad (J = J_2,\ \omega^2 = H_0^2 - \rho_0) \qquad (1.6)$$

Three cases should be examined: (1) $\omega^2 > 0$, i.e., either $\rho_0 < 0$ or $0 < \rho_0 < H_0^2$; (2) $\omega^2 < 0$, i.e., $\rho_0 > H_0^2$; (3) $\omega = 0$. For the first case, the solution is given by

$$u = H_0 y,\ v = \frac{\omega \cos \tau}{\varkappa + \sin \tau} y,\ \varphi = \frac{\varkappa \omega^2}{2(\varkappa + \sin \tau)} y^2, \\ \rho = \frac{\varkappa\omega^2}{\varkappa + \sin \tau},\ H = H_0, \qquad (1.7) \\ (\tau = \omega t + \varepsilon,\ |\varkappa| = |\gamma/\alpha| < 1,\ \alpha^2 = \gamma^2 + \omega^2)$$

It is apparent that when the normal velocity component is determined, the addition of a compensating background is equivalent to amplification of the magnetic field and, conversely, the addition of a background of the same sign is equivalent to attenuation of the magnetic field—the result is entirely natural (attenuation or amplification of E for a fixed H). The nature of the solution and the form of the trajectories are the same as those of solution (3.15) in [6] when $\rho_0 = 0$.

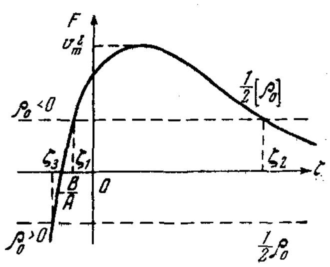

Fig. 2

In the second case, solving (1.6), we obtain

$$J^2 = \frac{\Omega^2}{\Omega^2 - \gamma^2} \frac{1 - \gamma^2 \operatorname{ch}^2 \tau + \Omega^2 \operatorname{sh}^2 \tau}{(\operatorname{sh} \tau - \varkappa)^2}, \\ (\varkappa = \gamma / \sqrt{\Omega^2 - \gamma^2},\ \omega = i\Omega,\ \tau = \Omega t + \varepsilon,\ \Omega^2 - \gamma^2 > 0) \\ J = \frac{\Omega}{1 - \Omega e^{-\tau}} \qquad (\Omega = \gamma). \qquad (1.8)$$

The form of the function $J(\tau)$ when $\Omega^2 - \gamma^2 > 0$ is shown in Fig. 1. The remaining parameters of the flow behave in the same manner, suffering a discontinuity at the point $\operatorname{sh} \tau = \varkappa$ and monotonically decreasing as $|\tau| \to \infty$; here

$$J_1 = H_0,\ J_2 \to \Omega,\ J_4 \to {}^1/_2 (\Omega^2 + H_0^2),\ J_5 \to 0.$$

In the third case, Eq. (1.6) agrees with the equation that describes the change in velocity with respect to time in electrostatic flow between parallel planes and at $\rho_0 = 0$ [6]. The solution is given by the curves in Fig. 3 [6]—the only difference is that the curve for J_4 must be raised by H_0^2.

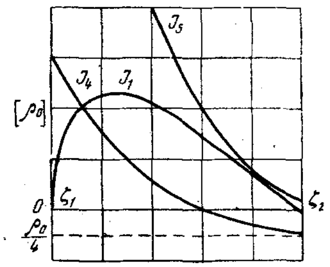

Fig. 3

5°. 4. Let us consider an electrostatic solution of this form. System (S/H) is then

$$J' + J^2 = 2J_4, \qquad J_5' + 2JJ_5 = 0, \\ 4J_4 = J_5 + \rho_0 \quad (J = J_1). \qquad (1.9)$$

Eliminating J_4 and J_5, we obtain

$$J'' + 4JJ' + 2J^3 - \rho_0 J = 0. \qquad (1.10)$$

Solution (1.10) is given by the expression

$$J^2 = 2(A\zeta + B)e^{-2\zeta} + {}^1/_2\rho_0, \qquad (1.11)$$

where t and ζ are linked by the relation

$$t = \int \frac{d\zeta}{\sqrt{2(A\zeta + B)e^{-2\zeta} + {}^1/_2\rho_0}}. \qquad (1.12)$$

It is not difficult to see that $J_5(\zeta)$ is specified by the same curve as in Fig. 4 [6], and $J_4(\zeta)$ is obtained when the corresponding curve in Fig. 4 is raised by $\rho_0/4$. Note, however, that relation (1.12) differs from $t = t(\zeta)$, which was derived in [6]. Solution (1.11) is not

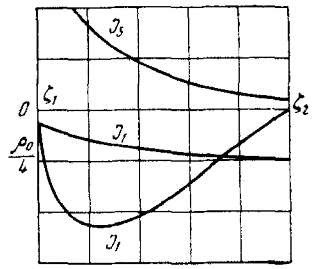

Fig. 4

valid at all values of ζ but only at those values for which $J^2 \geq 0$. Figure 2 illustrates this. When $\rho_0 > 0$ (background of the same sign as the charge of the beam particles), the solution makes sense over the interval $\zeta_3 \leq \zeta < \infty$. The nature of the flow is the same as when $\rho_0 = 0$ [6]—the only difference is that $J_1 \to \sqrt{{}^1/_2\rho_0}$, $J_4 \to {}^1/_4\rho_0$, $J_5 \to 0$. When $\rho_0 < 0$ (compensating background), the solution is valid when $\zeta_1 \leq \zeta \leq \zeta_2$; the velocity vanishes at the ends of the interval (Fig. 3). When $J_4 = 0$, the beam density and the background density are equal. Later $J_5 < |\rho_0|$, $J_4 < 0$. When $\zeta = \zeta_2$, a uniform distribution of the space charge with density lower than the background density is established in a cylindrical diode. A solution of type (1.11) can describe the process of monotonic decrease in density from this state, too. In fact, for this the following inequality must hold

$$v_m^2 = A \exp\left(\frac{2B}{A} - 1\right) > \frac{|\rho_0|}{2}. \tag{1.13}$$

For simplicity, let us assume that A = B. Besides (1.13), by convention $J_5 < |\rho_0|$. Hence, we find that A must satisfy the inequalities

$$\frac{1}{e} < \frac{A}{|\rho_0|} < 1.$$

Here, the solution has the form shown in Fig. 4. It can be shown that solutions of the form of (1.11) with $J_5(\zeta_2) > |\rho_0|$ do not exist.

10°. Solution of system (S/H) leads to the result

$$u = \frac{a + f'x}{f}, \quad v = \frac{b + g'y}{g}, \quad \varphi = \frac{f''x^2}{2f} + \frac{g''y^2}{2g}, \quad \rho = \frac{R_0}{fg} \tag{1.14}$$

and for $f(t)$ and $g(t)$

$$f''g + g''f = R_0 + \rho_0 fg.$$

Thus, one of the functions $f(t)$ and $g(t)$ can be assigned arbitrarily; a, b, R_0 are arbitrary constants. When $g \equiv 0$, we have

$$u = \frac{a + f'x}{f}, \quad \varphi = \frac{f''x^2}{2f}, \quad \rho = \frac{R_0}{f}. \tag{1.15}$$

For $f(t)$, we obtain the equation

$$f'' - \rho_0 f = R_0.$$

The case of a compensating background $\rho_0 = -\omega^2 < 0$ is more interesting. Here

$$f = A \sin\tau + R_0/\omega^2 \quad (\tau = \omega t + \delta;\ A, \delta = \text{const}).$$

By requiring that the solution be continuous, we arrive at the inequalities

$$A > 0, \qquad |R_0| > A\omega^2.$$

When these conditions are satisfied, solution (1.15) takes the form

$$u = \frac{a + \omega x \cos\tau}{\varkappa + \sin\tau}, \quad \varphi = -\frac{\omega^2 x^2 \sin\tau}{2(\varkappa + \sin\tau)}, \quad \rho = \frac{R_0/A}{\varkappa + \sin\tau}$$

$$(|\varkappa| = |R_0(A\omega^2)^{-1}| > 1;\ \varkappa > 0,\ R_0 > 0;\ \varkappa < 0,\ R_0 < 0) \tag{1.16}$$

The time part of functions (1.16) is represented in Fig. 5 ($I_1 = J_1/\omega$, I_4/ω^2, $I_5 = AJ_5/R_0$); these are periodic functions with period $T = 2\pi/\omega$, which is determined by the background density.

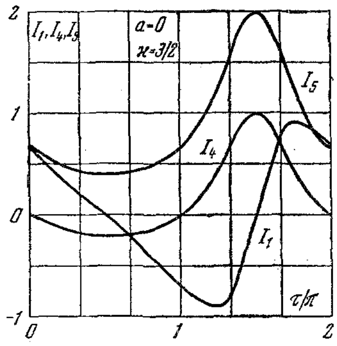

Fig. 5

Everything that was said in [6] about the possibility of extending two-dimensional solutions to the three-dimensional case remains in force. Of the solutions with undeformed equipotential surfaces examined in [6] in the presence of a background, one remains: the invariant solution for the subgroup H $\langle X_1 + X_2, X_3, X_4\rangle$, which describes the electrostatic flow between concentric spheres

$$v_r = rJ_1(t), \quad \varphi = r^2 J_4(t), \quad \rho = J_5(t). \tag{1.17}$$

System (S/H) for (1.7):

$$J' + J^2 = 2J_4, \quad J_5' + 3JJ_5 = 0$$
$$6J_4 = J_5 + \rho_0 \quad (J = J_1). \tag{1.18}$$

By eliminating J_4 and J_5, we obtain

$$J'' + 5JJ' + 3J^3 - \rho_0 J = 0. \tag{1.19}$$

Solution (1.19) has the form

$$J^2 = 2e^{-2\zeta}(Ae^{-\zeta} + B) + {}^1/_3\rho_0, \tag{1.20}$$

where t and ζ are linked by the relation

$$t = \int \frac{d\zeta}{\sqrt{2e^{-2\zeta}(Ae^{-\zeta} + B) + {}^1/_3\rho_0}}. \tag{1.21}$$

When B = 0, expression (1.21) is written as follows:

$$\frac{1}{3}\left(\frac{2}{\rho_0}\right)^{1/2} \operatorname{ar\,sh}\left(\frac{\sqrt{\rho_0}}{A} e^{3/2\zeta}\right) = t + c, \qquad (\rho_0 > 0),$$
$$\frac{1}{3}\left(\frac{2}{|\rho_0|}\right)^{1/2} \arcsin\left(\frac{\sqrt{|\rho_0|}}{A} e^{3/2\zeta}\right) = t + c, \qquad (\rho_0 < 0).$$

Solution (1.20) when $\rho_0 < 0$, just as (1.11) has meaning over the interval $\zeta_1 \leq \zeta \leq \zeta_2$, which is defined by the requirement $J^2 \geq 0$. Functions J_4 and J_5, as functions of ζ, are specified by the same curves as in [6], but the curve for J_4 is raised by $\rho_0/3$. The nature of the flows described by solution (1.20) is the same as that for solution (1.11).

It should be noted that everything said in [6] about invariant solutions of rank 2 is valid here, too.

The fact that, in the presence of a background, the equations of an electrostatic beam permit transformation of coordinates, velocities, and potential with arbitrary functions of time makes it possible to formulate some nonstationary transform in accordance with any stationary solution [7]; the latter solution will not be similar to any of the nonstationary invariant solutions examined above. Solutions that describe certain processes in a plane diode are the simplest to interpret.

§2. Invariant solutions of the equations of a multicomponent beam. When the beam is formed by particles of several kinds, the equations of motion and of current conservation must be written out for each component. The interaction of particles with different specific charges is taken into account by means of the total space-charge density in the right side of the Poisson equation

$$(S)\qquad \begin{gathered}\frac{\partial v_{(s)}{}^{i}}{\partial t}+v_{(s)}{}^{k}\left(\frac{\partial v_{(s)}{}^{i}}{\partial x^{k}}+\Gamma_{pk}^{i}v_{(s)}{}^{p}\right)=\\ =\alpha_{(s)}g^{il}\left(\frac{\partial\varphi}{\partial x^{l}}+\sqrt{g}\,e_{mnl}v_{(s)}{}^{m}H^{n}\right)\\ \frac{\partial\rho_{(s)}}{\partial t}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{i}}\left(\sqrt{g}\,g^{ik}(\rho v_{k})_{(s)}\right)=0,\\ \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{i}}\left(\sqrt{g}\,g^{ik}\frac{\partial\varphi}{\partial x^{k}}\right)=\sum_{s}\rho_{(s)}\end{gathered}$$

Here $v_{(s)}{}^{(i)}$, $\rho_{(s)}$ are the velocity components and the space-charge density of the s-component; $\alpha_{(s)}$ are constants that arise in dedimensionalization and are due to the different value and sign of the specific charge η of the particles of the different components. The principal group of system (S) is determined by operators (2.1) in [6]—the only difference is that now $\rho_{(s)}\partial/\partial\rho_{(s)}$, $V_{(s)}\nabla_{V(s)}$ should be understood as $\rho\partial/\partial\rho$, $V\nabla_V$; summation is performed with respect to s; rotation in plane u, v, for example, of a velocity space u, v, w $(-v\partial/\partial u+u\partial/\partial v)$ is replaced by rotation by the same angle in planes $u_{(s)}$, $v_{(s)}$ of the velocity spaces $u_{(s)}$, $v_{(s)}$, $w_{(s)}$ of each of the components $(-v_{(s)}\partial/\partial u_{(s)}+u_{(s)}\partial/\partial v_{(s)})$. The equations of an electrostatic beam lose their exclusivity, losing the additional—as compared with the case of $H\neq 0$—transformations with arbitrary functions of time.

The optimal systems of subgroups (2.6) and (2.7) in [6] can be used without any changes to construct all the essentially different H-solutions of ranks 1 and 2. But now the division of the subgroups of these systems into three classes does not hold: the operators that form the subgroups of the third class do not exist, while the subgroups of the second class give essentially different H-solutions for any H. The type of solution is established by means of the table in [6]. The velocity vector and space-charge density of all components have the same form (see columns of table for V and ρ). All that was said about three-dimensional flows [5, 6] remains in force.

If the velocities of all components are relativistic, then the results of paragraph 8 of [6] and [5] are similarly extended to the case of multicomponent beams. But if the velocities of only a few components are relativistic, then the principal group of such a system is less general than (8.4) in [6]: only expansions, rotations of the coordinates in space, and translations remain.

In [4–6] it was assumed that the velocity vector was a single-valued function. If V is an s-valued function (s > 1), then it can be assumed that we are dealing with s elementary monoenergetic beams. Here, all $\alpha_{(s)}$ in (S) are equal to unity. It is easy to see that the principal group of the obtained equations coincides with the principal group G_t of the equations of a monoenergetic beam [6] taking into account the above remark on the meaning of $\rho\partial/\partial\rho$, etc. Thus, all the H-solutions that were examined in [4–6] are possible for a multivelocity beam.

It is natural to expect that transformations with arbitrary functions of time that preserve the equations of a multivelocity electrostatic beam with an arbitrarily large s will also hold when $s\to\infty$, i.e., on moving to a description by means of the distribution function. It can be seen that Vlasov's equations actually permit these transformations. This makes it possible to obtain certain nonstationary solutions, if we know the stationary solutions of these equations [7].

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